広島大学総合科学部紀要Ⅳ理系編，第31巻，11 －123ページ，2005年12月

An example of convex heptagon with Heesch number one

Yoshio AGAOKA

Department of Mathematics, Faculty of Integrated Arts and Sciences
[illegible]*ima 739-8521 a a*
e-mail address: agaoka@mis.hiroshima-u.ac.jp

Abstract: We give an example of convex heptagon whose Heesch number is just equal to one, and among fourteen kinds of edge-to-edge coronas of this tile we present some of them, one of which admits a family of continuous deformations.

Key words: monohedral tiling, convex heptagon, coron [illegible] c n [illegible] r

1. In this paper we consider edge-to-edge monohedral tilings of a region of the Euclidean plane by convex polygons. A tiling is called *edge-to-edge* [illegible] the interior of an edge of any other polygon, and is called *monohedral* if the region of the plane is tiled by congruent or reflected copies of a prototile T. For definitions of se [illegible] ental concepts on tilings, see [18], [29].

It is well known that any triangle, or any quadrangle including the non-convex one can tile the whole plane monohedrally, but any convex [illegible] -gon with $n \geq$ [illegible] g. [illegible] howed that there are three classes of convex hexagons that can tile the plane monohedrally, and as for the pentagon, we now know 14 classes of convex pentagon admitting a monohedral tiling. But the classification for the pentagonal case is not completed yet. For the detailed results and the complicated history of the classification of monohedral tilings by convex polygons, see the references listed up at the end of this paper. The present situation is extensively summarized in the excellent book Grünbaum-Shephard [18; [illegible]], [28], [32].) A similar problem for the spherical case is also interesting and is considered in the papers [6]~[10], [31], [33]~[3], etc.

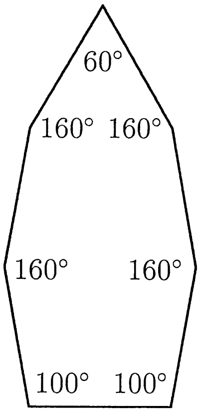
2. On the other hand, concerning local tilings of the plane, the concept "Heesch number" was introduced in order to measure the ability of a tile to tile the plane to what extent. To explain this concept, we first consider a "corona" of a tile. A *corona* is a tiling around one tile T by congruent or reflected copies of T such that no point of T cannot be visible from the exterior in the [illegible] minimal condition: If we drop one tile from this local tiling around T, then a part of T is visible from the exterior. Starting from one tile T, we construct a corona around T, and next we construct a second corona around the first corona, satisfying the same conditions as above, where we replace T with the first corona in this case. We continue to repeat this procedure until we cannot construct a new corona. Among several construction of

2005年10月 1 日受付；2005年11月 1 日受理

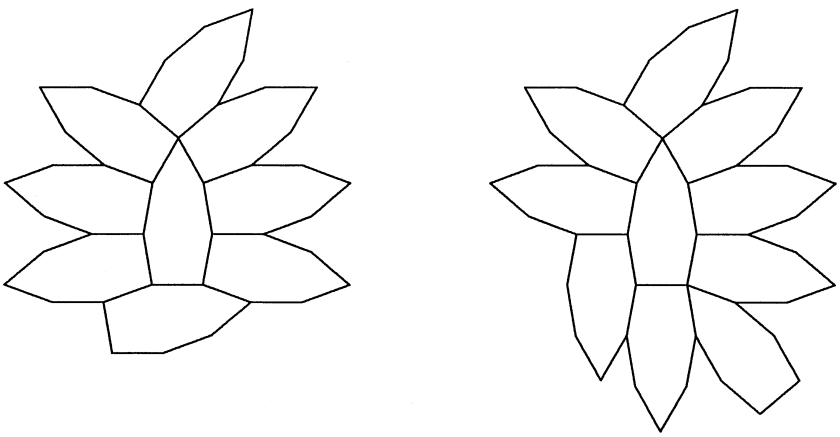
=0=. Sevral vi Snti si v. si gl 0v=xvnSpl 0enumberr angi nvF nbi no t [4] n, SnT
T T admits a monohedral tiling of the pT = con voex hpt ovxhoa pgo - gons with n T = 0. Several interesting examples and results on the Heesch number are given in Fontaine [14], S(((((
ete te t t e t e e- - - - - onditions on a tile or a ", or the property "edge-to-edge".

=0=. S. ev0Sr al ei0Snt 0sanl eg0x em Span T and so the Heesch number is finite for any convex heptagon. Keeping this fact in mind, Morimoto [22] offered a conjecture that $H(T) = 0$ for a o e e ago a e convex heptagon T cannot be surrounded by congruent or reflected copies of T. Th s this paper is to give a counter-example to this conjecture. We give an example of a convex he T with $H(T) = 1$. The results are summarized in the following form:

The Heesch number of the following equilate al e e ta (bamboo shoot) is one.

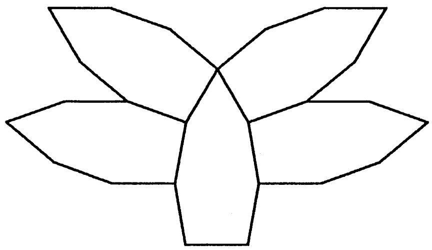


We can construct a corona of this ti clidean transformations, all of which are necessarily edge-to-edge. Among then we give two examples here:



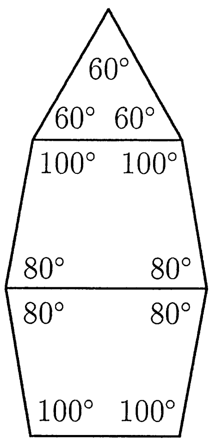
p l e t o h s ecnejcl u tier i .oWl eg .rea TT T T

”, o r thepy “dtgy - .““d T T T T T T T
and we here only sketch an outline of the proof. First, it is an easy task to check that the above figures actually give coronas of the central heptagon. In addition, if a vertex of surrounding tiles lies on the interior of an edge of the central heptagon, then an exterior angle 20° or 80° necessarily appears in some place, and so we cannot construct a complete corona. Hence any first corona of this tile must be edge-to-edge. By an elementary argument we can also show that any coronas necessarily contain the following partial tiling:



And we can fill the upper exterior part of this figure essentially in a unique way up to an axial reflection. Also there are essentially eight ways to fill the lower exterior part, two of which are axially symmetric. By considering all combinations up and below, we know that there are exactly [illegible] idean transformations. For all these coronas an angle 40° appears as an exterior angle, and so we cannot construct a second corona in any case, which shows that the Heesch number of this tile is exactly one.

The most fundamental part of the proof is to ensure the existence of an equilateral convex he he [illegible] angles and lengths of edges cannot be independently chosen uires many calculations. But in this c s c n si s the existence by drawing two line segments in the tile as follows:

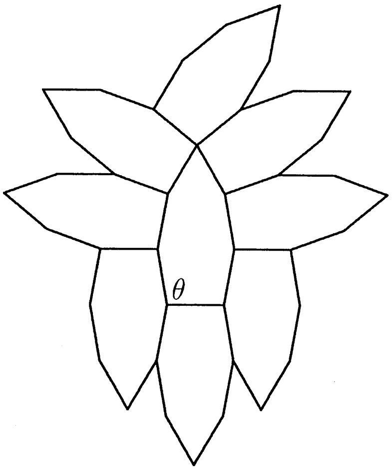


By this figure we know that the desi [illegible] congruent tr e o and one equilateral triangle. And thus we obtain Theorem. q.e.d.

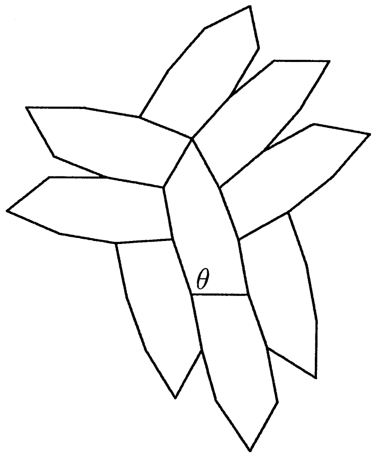
wms i i i mi i n

- Consider similar problems not only i ine, conformal or projective geometry, as Bowers-Stephenson [5] and Benoist [2] have done.

Note that, concerning the second problem, there are infinitely many such heptagons. In fact the corona exhibited left below can be continuously deformed to the right one:



$\theta = 100°$



$\theta = 110°$

The angles of this deformed equilateral heptagon are given by 60°, $360° - 2\theta$, $\theta + 60°$, θ, $300° - 2\theta$, $\theta + 60°$, $\theta + 60°$ ($90° < \theta < 120°$) counterclockwise from the top. The existence of such a heptagon is ensured by the same method as in Theorem. In this case the heptagon is constructed from one equilateral triangle and two trapezoids with interior angles $300° - 2\theta$, $2\theta - 120°$, $180° - \theta$, θ. In case $\theta = 120°$ this heptagon degenerates to a trapezoid a d i a e $\theta = 90°$ it reduces to a hexagon of Type 1, Type 2 listed in [18; p.494]. The case $\theta = 110°$ is interesting among this family of heptagons because a corona of special type appears in this case.

References

[1] O. Bagina, *Tiling the plane with congruent equilateral convex pentagons*, J. Comb. Theory A, **105** (2004), 221-232.

[2] Y. Benoist, *Pavages du plan*, in Pavages (eds. N. Berline, C. Sabbah), 1-48, Édition de l'École Polytechnique, Palaiseau, 2004.

[3] M. Berger, *Convexity*, Amer. Math. Monthly, **97** (1990), 650-678.

[4] B. Bollobás, *Filling the plane with congruent convex hexagons without overlapping*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **6** (1963), 117-123.

[5] P. L. Bowers a . t e so *A "regular" pentagonal tiling of the plane*, Co o o Dynamics, **1** (1997), 58-86.

[6] A. M. R. Azevedo Breda, *A class of l s of* ^{2}, Geom. Dedicata, **44** (1992), 241-253.

[33] Y. Ueno and Y. Agaoka, *Tilings of the 2-dimensional sphere on r en ri ht triangles*, Mem. Fac. Integrated Arts and Sci., Hiroshima Univ. Ser.IV, **22** (1996), 1–23.

[34] Y. Ueno and Y. Agaoka, [illegible]*gruent quadrangles*, Mem. Fac. Integrated Arts and Sci., Hiroshima Univ. S . **27** (2001), 135–144.

[35] Y. Ueno and Y. Agaoka, *Classification of tilings of the 2-dimensional sphere by congruent triangles*, Hiroshima Math. J., **32** (2002), 463–540.

[36] B. Wichmann, *The World of Patterns* (with CD-ROM), Worl i ti i, 01.